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Density distribution techniques and strained length methods for determination of finite strains

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Abstract—For a homogeneously deformed rock composed initially of an isotropic distribution of object shapes, finite strain may be determined from the correlation between the orientations of either two-dimensional or one-dimensional sample cuts and the frequencies with which they intersect marker objects. Mimran previously published an incorrect method for planar samples under the heading 'density distribution technique'. Methods are described by which the three-dimensional strain may be directly determined from six general samples, either linear or planar. Construction of two-dimensional ellipses as an intermediate step is unnecessary and enforces practical difficulties.

These methods may be simplified by use of samples parallel to known principal axes or planes of the finite strain. In this case the same large errors may arise from slight misorientation of samples as with other methods of strain measurement. A new quick method is proposed, combining linear and planar measurements of frequencies of intersected objects, which is thought to be the first method to circumvent a large part of the error from this error source. For example, if true $X:Z$ ratio is 9 : 1, and orientations in the XZ plane are misjudged by 8°, normal methods give 38% error where the new method gives, with care, an error of 1.9%. For methods of strain measurement such as are described here the concept of strain ellipsoid is unnecessarily limiting, and should be abandoned.

INTRODUCTION

THE PRINCIPLE of strain measurement methods under the heading of 'density-distribution technique' was introduced by Mimran (1976), from which the following quotation is taken (pp. 175–176): "The technique of finite strain analysis described here is based on indirect measurements of deformed strain indicators. In other words, no actual measurements of the dimensions of individual markers are required. The basic principle is that the longer the dimension of an object in a certain orientation, the higher the probability of its being intersected by a plane perpendicular to this orientation". So by measuring the density of intersected objects on planes of different orientations it is possible to judge the relative 'length' of the strain ellipsoid in the direction perpendicular to each section plane, and from that to reconstruct the complete strain ellipsoid.

The principle of such strain measurement techniques is a good one, as it is applicable to objects of unknown initial shape (provided they were isotropically arranged in the first instance), and statistical errors are minimized by counting over large numbers of intersected objects. Unfortunately, the development of the technique given by Mimran (1976) is spurious on account of a simplistic extrapolation of two-dimensional geometry into three dimensions, which leads to very large errors in practice. This paper attempts to explain the geometry necessary to re-establish density-distribution techniques on a correct basis, and then considers methods of application.

All the methods specifically mentioned here depend upon two assumptions: that the initial distribution of shapes was isotropic prior to deformation; and that the objects counted on sample lines or planes have deformed homogeneously with their matrix. More widely relevant observations arise from the discussion, particularly concerning errors when principal planes are assumed, and concerning unnecessary constraints which an intermediate step of evaluating two-dimensional ellipses may impose on determinations of strain in three dimensions.

THE DIMENSION OF AN ELLIPSOID PERPENDICULAR TO A PLANE

For a density distribution technique, a comparison is made of several differently oriented sample planes. These are assumed to have contained the same density of intersections of marker objects (e.g. ooids, calcispheres, feldspar phenocrysts, sand grains) in their initial states. The initial dimensions of all these sample planes are represented by the circular sections of an initial sphere of unit radius having the (unknown) initial orientations of these samples. Deformation produces the observed sample orientations. It also changes the size of each sample to an extent represented by the ratio of the area of a parallel section through the finite strain ellipsoid to the area of a section of the initial sphere. Assuming each sample plane intersects the same objects as it did before deformation, the ratio of observed density to initial density is given by the inverse of this ratio of areas. The initial density is unknown. Therefore, the absolute area of each section through the finite strain ellipsoid cannot be determined. However, the ratio of any two section areas is determinable, being given by the inverse ratio of the corresponding sample densities.

For any section cut through the centre of an ellipsoid, the volume of the ellipsoid is given by $4/3\ Ah$, where 'A' is the area of the section and 'h' is the 'perpendicular height'. Perpendicular height means here the dis-

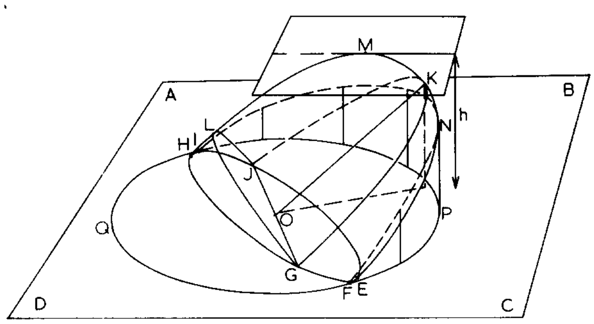

Fig. 1. Sketch of half an ellipsoid to illustrate terms used in the text. Line *OK* represents the longest (*X*) axis, rising to the right and slightly away from the observer. The (*YZ*) plane perpendicular to this axis is represented by *GLJ*. The horizontal plane *ABCD* through the centre of the ellipsoid, *O*, cuts the ellipsoid surface along the section ellipse *EFGHIJ*, of area *A*. The horizontal plane tangent to the ellipsoid at its highest point, *M*, is at height h above *ABCD*. The dashed line *FNI* joins all points where the ellipsoid surface contains the vertical direction. This line, which does not lie in a plane, is the ellipsoid edge when viewed vertically. It projects vertically onto *ABCD* as the projection ellipse *FPIQ*. Lines on *ABCD* which are vertical projections of vertical tangent planes to the ellipsoid are tangents to this projection ellipse. Perpendicular lines from *O* to such tangents are coincident with the lines perpendicular to the corresponding tangent planes of the ellipsoid in 3–*D* which have relative lengths estimated from areal density distributions on corresponding parallel sample planes which share the vertical linear direction.

tance separating the section plane from a parallel tangent plane to the ellipsoid (Fig. 1). Therefore, for a pair of sections through the same ellipsoid, the ratio of their perpendicular heights is equal to the inverse ratio of their areas. This is equal to the ratio of intersected object densities of samples which they represent. A series of measured sample densities may, therefore, be considered as a measure of the distance from the centre of the ellipsoid to a corresponding series of bounding tangent planes. Their absolute distances are not known, but relative distances are sufficient to determine the orientations and the ratios between the lengths of the axes of the finite strain ellipsoid.

SECTION AND PROJECTION ELLIPSES OF AN ELLIPSOID

Lengths of lines from the centre of a strain ellipsoid to its surface are by definition representative of strained lengths which were initially equal. When a specimen is examined on a sample plane and relative real lengths, of whatever kind, in different directions in the plane are compared, the representative lengths in the strain ellipsoid are those lying in a section through the ellipsoid centre parallel to the sample. This 'strain ellipse' I call a 'section ellipse' of the strain ellipsoid. When several strain ellipses are used to reconstruct the strain ellipsoid, it is sections of the ellipsoid which are combined.

Suppose a density distribution technique is applied to a family of sample planes sharing a common linear direction. This will give a relative measure of perpendicular distances from the centre of the strain ellipsoid to a family of tangent planes which share the common direction. This situation can be better viewed by projecting along the common direction onto a perpendicular plane (Fig. 1). The lines which are projections of the tangent planes are now tangent to an ellipse which is the projection of the edge of the strain ellipsoid viewed down the projection direction. This ellipse I call the 'projection ellipse' of the strain ellipsoid. It is not generally similar in orientation or ellipticity to the strain ellipse of the plane in which it lies.

The two ellipses become identical in the case of a principal plane, but errors from slight deviation from principal planes are even larger in strains calculated on the basis of identity of section and projection ellipses than in strains calculated on the assumption that either is an exact principal plane (see later discussion).

EQUATIONS OF A SECTION ELLIPSE AND A PROJECTION ELLIPSE

To illustrate the relationship between section and projection ellipses, consider orthogonal reference directions of which two lie in the plane to be considered, but which are not generally related to the principal axes of the ellipsoid. The third reference direction will be the direction of projection for the projection ellipse. Take the following general equation and specific example:

$$fx^2 + gy^2 + hz^2 + jyz + kzx + lxy = 1$$

$$x^2 + y^2 + 4z^2 + 3yz + 3zx + xy = 1.$$

The section ellipse on the plane parallel to x and y axes is the locus of points obeying the ellipsoid equation and for which $z = 0$. Hence:

$$fx^2 + gy^2 + lxy = 1$$

$$x^2 + y^2 + xy = 1.$$

To find the projection ellipse consider the ellipsoid equation written as an equation in z:

$$hz^2 + (jy + kx)z + (fx^2 + gy^2 + lxy - 1) = 0.$$

Any chosen value of x and y can be taken to specify a line parallel to z in space, which will penetrate the ellipsoid if any real solution giving z exists for that x and y. Such a line will generally penetrate the ellipsoid twice, corresponding to two possible values for z. The ellipsoid edge as viewed down z is where the two z values become coincident. For this situation, the ellipsoid equation can be written in the form:

$$(h^{1/2}z + (fx^2 + gy^2 + lxy - 1)^{1/2})^2 = 0.$$

This gives two identical solutions of z (of no further interest), but this form is only correct for the ellipsoid provided:

$$2h^{1/2}(fx^2 + gy^2 + lxy - 1)^{1/2} = (jy + kx).$$

This equation linking x and y must be true for the ellipsoid edge and for any projection of it down z. Squaring both sides and rearranging, it gives the following equation of the projection ellipse in normal form:

$$(f - k^2/4h)x^2 + (g - j^2/4h)y^2 + (l - 2jk/4h)xy = 1.$$

For the example this gives:

$$(7/16)x^2 + (7/16)y^2 - (1/8)xy = 1.$$

This example is chosen to highlight the differences between section and projection ellipse. In this case the long axis direction of one actually corresponds to the short axis direction of the other. In general they will lie oblique to each other. The semiaxis ratios are for this section ellipse $\sqrt{3} : 1$, and for the projection ellipse $2 : \sqrt{3}$.

ON DETERMINATION AND COMBINATION OF PROJECTION ELLIPSES

Although not advised as a method of determining the three-dimensional strain state of a rock, one way would be to determine projection ellipses of the strain ellipsoid and then combine them to give the strain ellipsoid.

The only method of determination of a projection ellipse apparent to this author is by the measurement of intersected object density on each of three different planes sharing the projection direction. In theory, use of two such measurements plus knowledge of the projection ellipse orientation would suffice, but there is no apparent method by which this orientation may be determined. In particular, the long axis of the projection ellipse will not generally be parallel either to the preferred elongation direction of objects in the plane of the ellipse (i.e. the major axis of the parallel section ellipse) or the trace of the XY plane of the strain ellipsoid. Three projection ellipses may be combined to give the strain ellipsoid, but the method is quite different from those discussed by Ramsay (1967, pp. 142–149 and 199–200) and by Roberts & Siddans (1971), for combination of section ellipses. The necessary algebra for determination of projection ellipses and for their combination, is outlined in Appendix B.

Mimran (1976) failed to distinguish between projection and section ellipses of the strain ellipsoid. Believing them to be the same thing, he naturally suggested use of the most easily available information. Specifically, he constructed ellipses from two lengths, measured by object densities, plus the preferred object elongation direction. As this combines lengths belonging to the projection ellipse with the orientation of the section ellipse, the result is neither a good projection nor a good section of the strain ellipsoid. The ellipsoid produced by combining these as if they were sections, bears no sensible relationship to the three dimensional strain.

Mimran tackled a practical matter not dealt with in detail here. The mathematics of this paper assumes a two-dimensional sample, such as a cut surface. If a thin section is used in transmitted light, some correction may be necessary for sample thickness. Such a sample may be treated as two-dimensional if the objects are an order of magnitude or more greater in size than the section thickness. The method will fail if objects are minute relative to section thickness, because in this case the number of objects per unit volume would be counted in every case, with no variation with sample orientation. For intermediate object sizes, say within one order of magnitude either way of sample thickness, a correction factor should be used.

There is an important methodological difference between projection ellipse and section ellipse techniques. Section ellipses (strain ellipses) may be determined in diverse ways, and each parallels a real sample plane of study. Their combination to give the full three-dimensional strain is often a distinct synthetic final step in the procedure of strain determination. Projection ellipses, in contrast, only arise from object intersection density measurements, which could be combined to give the three-dimensional strain directly. Their only use is the calculation of this 3-D strain. Calculation of projection ellipses as a discrete procedural step is therefore without justification. Furthermore, such a step enforces unnecessary practical constraints. The sample planes must then lie in groups of three sharing a common direction. To reduce the number of sample planes from nine to six, yet still determine three projection ellipses, further requires that three of the sample planes each contain two of the projection directions. These constraints are totally unnecessary, as the 3–D strain may be established from any six general sample planes. If overdetermination is intended this may also be done by using additional sampling orientations, without requiring any thought of common directions (see Methods section, below).

AREAL DENSITY DISTRIBUTION METHODS

General method

Use sample planes in six different orientations, obtaining a value for the intersected object density on each, plus its orientation. Calculate the three-dimensional strain directly.

For good results the six planes should each be approximately equally spaced around a sphere. Imagine cubic-close-packed spheres. Around a central sphere there are twelve neighbours, one each end of six axial directions, with an interaxial angle of 60°. The best arrangement of sample planes is with their poles near to this regular pattern of axial directions. Potential errors in the final strain values depend on the arrangement of orientations used, and are not easily calculated.

For each sample plane, determine its orientation accurately. This paper assumes that this orientation is given as strike (α) and dip (β) (see Appendix A), but suitable algebra could be devised for other specifications without difficulty. Measure the overall density of objects intersected per unit area (d). Evaluate the coefficients of the following equation:

$$(\sin^2\alpha \sin^2\beta/d^2)t_{11} + (\cos^2\alpha \sin^2\beta/d^2)t_{22} + (\cos^2\beta/d^2)t_{33} + (-2\sin\alpha\cos\alpha\sin^2\beta/d^2)t_{12} + (2\sin\alpha\sin\beta\cos\beta/d^2)t_{13} + (-2\cos\alpha\sin\beta\cos\beta/d^2)t_{23} = \frac{1}{C^2}.$$

Take the equations so formed, one for each sample plane, as a simultaneous set, and solve for t_{11}, t_{22}, t_{33}, t_{12}, t_{13}, t_{23}. (These are overdetermined if more than six samples are used.) Set up a symmetric matrix **T** of these elements, and eliminate '*C*' by enforcing: det **T** = 1. Find the principal quadratic extensions λ_1, λ_2, λ_3 by solving the equation: det $(\mathbf{T}-\lambda\mathbf{I}) = 0$, where **I** is the 'identity' or 'unit' matrix.

For each quadratic extension λ, find the direction cosines of its principal axial direction (the components of vector **x**) by solving $\mathbf{Tx} = \lambda\mathbf{x}$ for the ratio of these components, and then scaling them so that their sum of squares is 1.

For details see Appendix C.

Principal planes method

Assuming the principal strain axes are known, find a value for the intersected object density on samples parallel to each principal plane. These values are in the ratio of the axial lengths of the strain ellipsoid. This approximation depends on how well the orientation of the principal planes can be defined. It is quick. For an alternative, more accurate, quick method, and for a discussion of errors, see later sections. Note here that errors arising from bad definition of principal planes are in general no better and no worse than for other methods.

LINEAR INTERSECTION LENGTH METHODS

An alternative to measuring the density of 2–*D* object intersections on sample planes is to measure the frequency of 1–*D* object intersections along sample linear directions. It is well in the context of this paper, to be explicit about differences and similarities in the two approaches.

The basis of the linear intersection length technique is to trace along a sample line of known orientation on a plane of study, and to record the number of objects passed through, or simply the number of boundaries crossed, in a certain distance. By using a series of parallel lines and obtaining the overall average length per object, counting statistics can, as with the areal density distribution technique, be good, The overall intersection length per object is proportional to the length in that direction from the strain ellipsoid centre to its surface. This is a normal 'strained length' measurement, in contrast to the length to a parallel tangent plane given by the areal density distribution technique.

Combinations of linear intersection length measurements in three directions on a plane establish the strain ratio and orientation of a normal strain (section) ellipse. The ellipse given by combinations of areal density distribution measurements, from planes sharing a common projection direction, was a projection ellipse of the strain ellipsoid.

Three general plane orientations are sufficient to give six general, independent, measures of linear intersection length from which the full 3-*D* strain may be calculated. With areal density distribution measurements, six general sample planes were needed. The number of planes is reduced to two, and to three, respectively, if the principal strain axes are known and sections cut along principal planes.

Using linear intersection lengths, the strain may be overdetermined without use of more than three planes of study, though additional planes may be more advantageous. With areal density distributions a new sample orientation was required for each additional measurement.

Several similarities in the two approaches exist. Each can define 3–*D* strain completely (except for volume change and body rotation) from six general measurements. Definition of 2–*D* ellipses as an intermediate step requires collection of redundant data, or requires that special common orientations are used for sampling, shared by pairs of ellipse planes. However, note that if ellipses are used, it is not just semi-axis ratio and orientation that are to be combined. These methods actually determine, rather than assume, values for the relative dimensions of the 2–*D* ellipses, because the proportionality of intersection densities or lengths exists fully in three dimensions. On the other hand, this is precisely why the intermediate step of determining 2–*D* ellipses is not of advantage. Direct analogy may be made between the following linear intersection length methods proposed and those for areal density distributions given earlier.

General method

Use planes of study in three orientations, counting intersection lengths in six general directions as near equally spaced (60°) around a sphere as possible. Take each length, *l*, to be a vector length in its direction, the local x^* direction, specified in terms of its strike (α), dip (β) and pitch (γ) (see Appendix A). If in the unstrained state each such vector **x** had length *c*, then the following equations can be applied:

$$\begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix} = \mathbf{x}^* = \mathbf{Rx}$$

$$\mathbf{x} = \mathbf{Sx}_0$$

$$\mathbf{x}_0^T\mathbf{x}_0 = c^2.$$

In these equations matrix **R** is specified in terms of α, β, γ (equivalent to **E** in Appendix A) and matrix **S** describes the strain (as in Appendix C). Defining a matrix $\mathbf{U} = (\mathbf{S}^{-1})^T\,\mathbf{S}^{-1}$, these equations combine to:

$$(l \;\; 0 \;\; 0)\mathbf{RUR}^{-1}\begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix} = c^2.$$

This is a linear equation in the elements of the symmetric matrix **U**, in terms of the common unknown '*c*', and the known coefficients, which are for each measured line its intersection length '*l*' and the elements of **R**. Six such simultaneous equations may be solved for the elements of **U**, and exactly the same procedure adopted as for the matrix **T** in Appendix C, except that the eigen values of **U** will be the inverse quadratic extensions of the strain.

Principal planes method

Cut planes (two would suffice) such that each principal strain direction lies in a plane of study, and measure the intersection lengths in these directions. These are in the ratio of the principal axial lengths of the strain ellipsoid. Errors are discussed below.

Strain ellipse method

Although not advised, 2–*D* strain ellipses may be determined on three planes. This requires nine linear directions, but may be reduced to six with three along intersection directions of the ellipses. These ellipses may then be combined as normal strain ellipses.

ERRORS IN 'PURE' PRINCIPAL PLANES METHODS

In the 'principal planes' methods described an important source of error is deviation of sample planes from the exact attitude of the principal planes of the strain. These errors are generally inherent in any method assuming known strain orientations, and some of the discussion below applies generally, not just to the methods given in this paper.

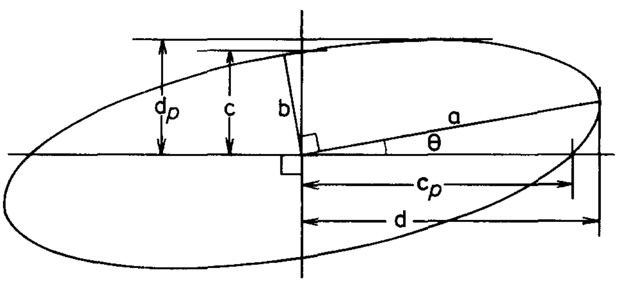

Fig. 2. Definition of lengths c, d, c_p, d_p, as used in the text, for sample lines at angle θ to the principal semiaxes of lengths a and b. Lengths c and c_p are intersection lengths proportional to lengths of intersected objects. Lengths d and d_p are projected lengths, proportional to densities of intersections of objects on perpendicular sample lines. Lengths c and d are the better estimates of b and a respectively.

Consider an ellipse (Fig. 2) with semiaxial lengths 'a' and 'b'. For a sample plane cut parallel to the short axis, any measurement of strained length, including intersection lengths, is proportional to 'b', while areal density distribution is an estimate of 'a'. The product of these two (ab) is a measure of ellipse area. Suppose the sample is misoriented by an angle θ. The product (cd) of intersection length 'c' and length 'd' to a parallel tangent is still the same measure (ab) of the ellipse area, whatever θ. Therefore, the fractional error in 'a' defined as $[(a-d)/a]$ and that in 'b' defined as $[(c-b)/c]$ are identical. Table 1 gives values of this fractional error for various angles and axial ratios.

Suppose measurements are made on a perpendicular pair of sample directions. The ratio of perpendicular lengths (d/d_p) to tangents, and the ratio of any strained lengths (c_p/c) is identical, because (cd) = (c_pd_p). Therefore the error in this as a measure of (a/b) is the same, whether the method used is areal density distributions, linear intersection lengths or any other measure of strained lengths.

However, this does not mean methods are equally good at defining different parts of an ellipse. It can be seen from Fig. 2 that 'd' from density distributions is a better estimate of the long axis 'a' than is the value 'c_p' obtained from a strained length method. On the other hand, for the short axis, strained length methods give 'c' which is a better estimate of 'b' than is 'd_p' obtained from density distributions. The second values in Table 1 for each value of θ and (a/b) are the fractional error when using the less suitable method for the axis being estimated, defined as $[(d_p-b)d_p] = [(a-c_p)/a]$. The errors given in Table 1 are plotted against θ in Fig. 3.

Table 1. Percentage errors in measured ellipse axial lengths (see text)

θ	(a/b) 1.5	2	3	4	6	9
1°	0.0085	0.011	0.14	0.14	0.015	0.015
	0.019	0.046	0.12	0.23	0.53	1.2
2°	0.034	0.046	0.054	0.057	0.059	0.060
	0.076	0.18	0.48	0.90	2.1	4.5
3°	0.076	0.10	0.12	0.13	0.13	0.14
	0.17	0.41	1.1	2.0	4.5	9.4
4°	0.14	0.18	0.22	0.23	0.24	0.24
	0.30	0.72	1.9	3.5	7.6	15.
6°	0.30	0.41	0.49	0.51	0.53	0.54
	0.68	1.6	4.1	7.3	15.	27.
8°	0.54	0.73	0.86	0.91	0.95	0.96
	1.2	2.8	6.9	12.	23.	37.
10°	0.84	1.1	1.3	1.4	1.5	1.5
	1.8	4.2	10.	17.	30.	46.
15°	1.9	2.5	3.0	3.2	3.3	3.4
	3.9	8.7	19.	29.	45.	60.
20°	3.3	4.5	5.3	5.6	5.9	6.0
	6.6	14.	28.	40.	56.	69.

Fig. 3. The percentage error, E, in the estimate of principal semiaxial length of an ellipsoid, which results from sample misorientation by θ degrees, for the indicated values of true axial ratio (a/b). The low errors (values circled) are for c and d in Fig. 2, defined by $E = (a-d)/a = (c-b)/c$. The high errors (dots only) are for c_p and d_p, defined as $E = (d_p-b)/b = (a-c_p)/a$. The values are given in Table 1.

Although the discussion above refers to a two-dimensional ellipse, the relationships are generally true for an ellipsoid in 3–*D*, and remain exact for its principal planes. Note at this point that a direct estimate of $X:Z$ ratio ($X>Y>Z$ extension) from two measurements, one of X and one of Z, is no better or worse than the result of using these values to calculate $[(X/Y).(Y/Z)]$ if the same Y value is used (even for an absurd value of Y). This point is significant in the context of methods which estimate lengths proportional to each of X, Y and Z, and not just ratios (X/Y), (Y/Z), (X/Z). It should not be thought that direct use of the 'XZ' plane to measure $X:Z$ necessarily conveys any advantage relative to indirect evaluation of this ratio, at least as far as the errors under discussion here are concerned.

Hybrid method

Cut just two sample planes, one parallel to the 'YZ' principal plane and the other parallel to the 'XZ' principal plane. Measure the areal density distributions on these two planes as an estimate of 'X' and 'Y'. Measure the linear intersection lengths along 'Y' and 'Z' as an estimate of these lengths on the 'YZ' plane. The resulting $X:Y:Z$ ratios have lower error arising from misorienting sample planes than for any 'pure' method, whether based on object intersections or not.

For consideration of errors assume that each sample plane deviates from its principal plane by an angle θ in the worst possible direction. The pole of 'YZ' deviates from 'X' towards 'Z'. The pole to the 'XZ' plane deviates from 'Y' towards 'X'. Because X is measured by density distribution, the error in the X value is small, almost all error in the $X:Y$ ratio being due to Y, as the shorter axis of the ellipse measured by density distribution. This error is therefore limited by the $X:Y$ ratio. Because the length Z is measured by intersection length, the error in this value is small, almost all error in the $Y:Z$ ratio being due to Y, as the longer axis of an ellipse measured by a strained length method. The error is therefore limited by the $Y:Z$ ratio. The errors in the resulting $X:Z$ ratio are shown in Table 2 and Fig. 4 for plane strain, for a variety of angular deviations and strain ratios. The top figure is the error in the hybrid method, due to angular deviation from principal planes as described, if the two 'Y' measurement directions are coincident (whether correct or not). As such, this figure represents the combined contribution to overall error of the errors in the X and Z estimates. The second figure is the combined error in the hybrid method resulting from angular deviation, assuming each 'Y' measurement direction is misoriented by the angle θ in its worst possible direction. In the case of the linear measurement this is towards Z, and in the areal density measurement towards X. This figure, where all four measurements are misoriented by θ in the worst possible way, is the maximum error in the hybrid method from this error source. In practice errors should lie between these two figures, given the deviation θ still in both 'X' and 'Z' directions. For comparison, the third figure given in Table 2 is the error due to these two

Table 2. Percentage errors in $X:Z$ ratio for plane strain resulting from deviations by angle θ from true orientations, for various true $X:Z$ ratios. (1) The combined contribution of X and Z measurement errors to the overall error in the hybrid method of this paper. (2) The maximum error for deviation by θ in all four measurements of the hybrid method. (3) The error from X and Z measurements in the determined $X:Z$ ratio by any pure method, whether based on object intersection measurements or not

	$X:Z$ Ratio					
Deviation	1.5	2	3	4	6	9
θ	0.068	0.091	0.11	0.11	0.12	0.12
2°	1.13	0.21	0.35	0.48	0.72	1.1
	0.11	0.23	0.54	0.96	2.1	4.1
	0.27	0.37	0.43	0.46	0.47	0.48
4°	0.51	0.85	1.4	1.9	2.8	4.2
	0.43	0.90	2.1	3.7	7.8	15.
	0.61	0.82	0.97	1.0	1.1	1.1
6°	1.2	1.9	3.1	4.2	6.2	9.0
	1.0	2.0	4.6	7.8	15.	27.
	1.1	1.5	1.7	1.8	1.9	1.9
8°	2.0	3.3	5.4	7.2	11.	15.
	1.7	3.5	7.8	13.	24.	38.
	1.7	2.3	2.7	2.8	2.9	3.0
10°	3.1	5.1	8.2	11.	16.	22.
	2.7	5.3	11.	18.	31.	47.
	2.4	3.2	3.8	4.1	4.2	4.3
12°	4.5	7.3	11.	15.	21.	29.
	3.8	7.5	15.	24.	38.	54.
	3.7	5.0	6.0	6.3	6.5	6.6
15°	6.8	11.	17.	22.	30.	39.
	5.8	11.	22.	32.	47.	62.

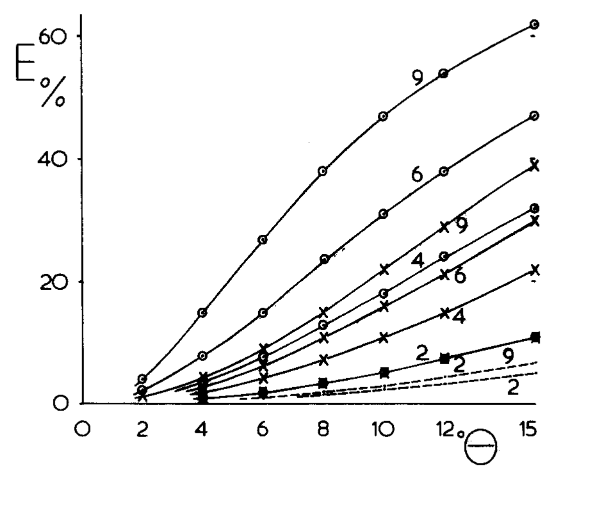

Fig. 4. The maximum percentage errors, E, in the estimate of principal strain ratio $X:Z$, for plane strain, which may result from misorientation of samples by θ degrees from the principal directions X and Z, for the true $X:Z$ ratios indicated. The high errors (values circled) are for normal methods of strain determination, including 'pure' density distribution methods of this paper. The intermediate errors (values crossed) are the maximum for the proposed hybrid density distribution method. These can occur only if both 'Y' samples are also in error (in different directions) by angle θ. The low errors (dashed lines, values dotted only) are the maximum in the hybrid method if a common 'Y' direction is used for the two 'Y' measurements. The advantage of the hybrid method is considerable for $X:Z$ ratios of 2 or more.

angular deviations in a directly measured $X:Z$ ratio by a 'pure' method, whether by areal density distributions, or by linear intersection lengths, or by any other method of strain estimation based on strained lengths.

Table 2 illustrates clearly how good the hybrid method described here is in terms of these errors. Perhaps it would be fairer to say that it illustrates how bad in general other methods of strain measurement are that assume that samples can be taken parallel to principal planes. Either way, the hybrid method, which is also potentially very quick, shows considerable advantage, and could be practically the most generally applicable and useful of the methods discussed in this paper.

CONCLUSIONS

General conclusions concerning use of the concept of 'strain ellipsoid' can now be drawn. In an early section it was argued that for areal density distributions the strain should be determined without recourse to 2–*D* projection ellipses as an intermediate step. Later it was argued that for relative strained length measurements, including intersection lengths, strain should again be determined without recourse to 2–*D* ellipses (strain ellipses in this case). Now that the various methods of strain determination have been described we can take this point further. It is not just 2–*D* ellipses which are unnecessary. The concept of 'strain ellipsoid' plays no necessary part in the determination of strain by any method, such as those here, which uses a homogeneous set of single measurements in known orientations. Sections or projections of the strain ellipsoid are therefore only worthwhile if a comparison with other strain measures is intended at the stage of a 2–*D* sample area. They are otherwise redundant.

As for intersection density methods, it has been shown that both areal density distributions and linear intersection lengths can be used correctly. The areal density distributions are practically more awkward but potentially more precise. If principal strain directions are known or assumed, both pure methods are greatly simplified, but errors are no better than with any other current method assuming these orientations. A hybrid method, using both areal density distribution and linear intersection length is proposed which can actually reduce such errors, while remaining quick and simple to perform.

REFERENCES

Mimran, Y. 1976. Strain determination using a density-distribution technique and its application to deformed Upper Cretaceous Dorset chalks. *Tectonophysics* **31**, 175–192.

Ramsay, J. G. 1967. *Folding and Fracturing of Rocks*. McGraw-Hill, New York.

Roberts, B. & Siddans, A.W.B. 1971. Fabric studies in the Llwyd Mawr ignimbrite, Caernarvonshire, North Wales. *Tectonophysics* **12**, 283–306.

APPENDIX A

Angular relationships between local axes and general reference axes in this paper are frequently used in matrix terms, and more specifically in terms of a suitable Eulerian angle set defined as in Fig. 5. The advantage of this system is that the three angles specified correspond to those conventionally measured by geologists. If X = north, Y = east, Z = down; then α = strike, and β = dip of the local $X'Y'$ plane, and γ is the pitch in this plane of X'.

If $\mathbf{x}' = \mathbf{E}\,\mathbf{x}$ relates local coordinates to those using the reference axes, and $\mathbf{R}_\alpha$, $\mathbf{R}_\beta$, $\mathbf{R}_\gamma$ are the individual rotations by α, β, γ; then:

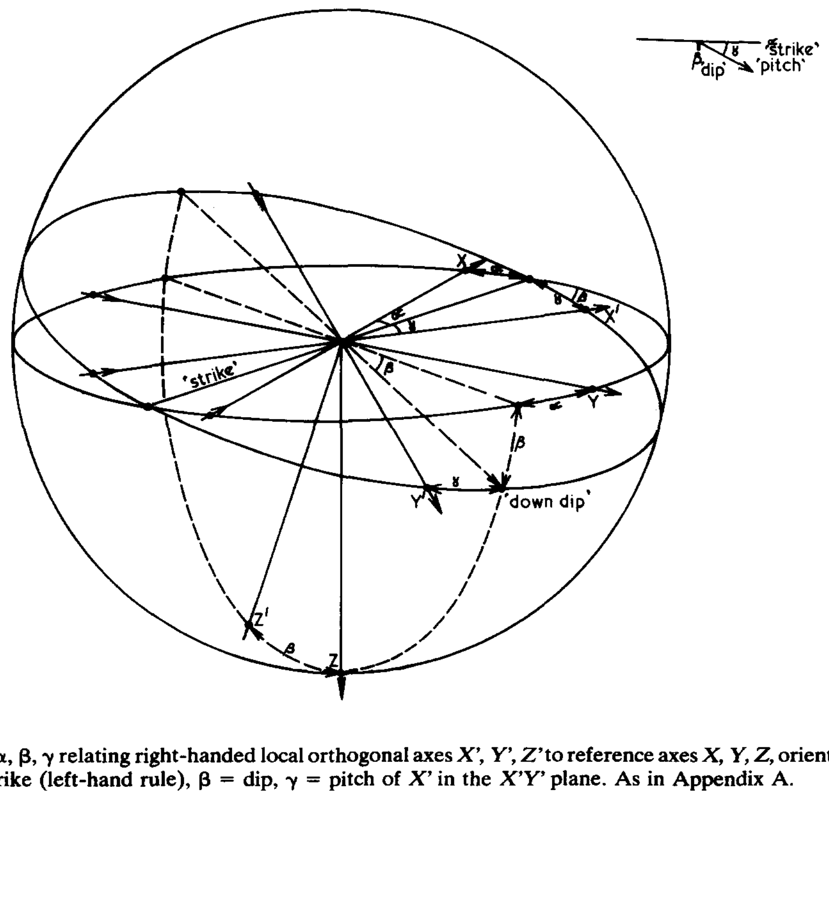

Fig. 5. Eulerian angles α, β, γ relating right-handed local orthogonal axes X', Y', Z' to reference axes X, Y, Z, oriented such that α = strike (left-hand rule), β = dip, γ = pitch of X' in the $X'Y'$ plane. As in Appendix A.

$$\mathbf{E} = \mathbf{R}_\gamma\mathbf{R}_\beta\mathbf{R}_\alpha = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (\cos\alpha\cos\gamma - \sin\alpha\cos\beta\sin\gamma) & (\sin\alpha\cos\gamma + \cos\alpha\cos\beta\sin\gamma) & (\sin\beta\sin\gamma) \\ (-\cos\alpha\sin\gamma - \sin\alpha\cos\beta\cos\gamma) & (-\sin\alpha\sin\gamma + \cos\alpha\cos\beta\cos\gamma) & (\sin\beta\cos\gamma) \\ (\sin\alpha\sin\beta) & (-\cos\alpha\sin\beta) & (\cos\beta) \end{pmatrix}$$

Note that:

$$\mathbf{E}^{-1} = \mathbf{R}_\alpha^{-1}\mathbf{R}_\beta^{-1}\mathbf{R}_\gamma^{-1} = (\mathbf{R}_\gamma\mathbf{R}_\beta\mathbf{R}_\alpha)^T = \mathbf{E}^T.$$

APPENDIX B

Part 1. Determination of a Projection Ellipse

An ellipse has semiaxes a and b, with one axis at an angle ϕ to a reference line. Lengths a, b and angle ϕ are unknowns (Fig. 6).

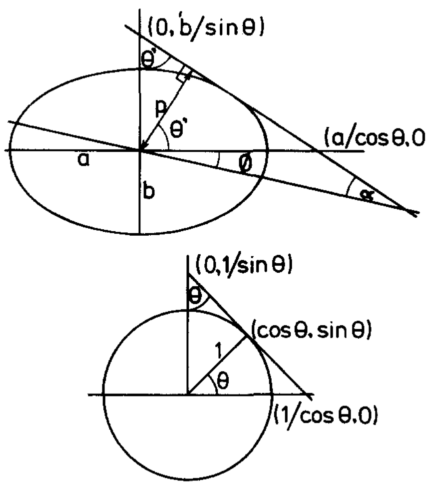

Fig. 6. Notation used in Appendix B for a tangent line at an angle α to a reference line itself at an angle ∅ to the principal axis of a strain ellipse.

Sample planes lie at angles α clockwise of the reference line, and density distributions on the sample planes are 'd'. The perpendicular length, p, is proportional to the density on the sample plane.

$$d = K.p.$$

In the unstrained state the tangent was perpendicular to a radius of unit length at an angle θ to the principal axis. The point of contact lay at (cosθ, sinθ). The tangent met the principal axes at (1/cosθ,0) and (0,1/sinθ). In the strained state these points became: (a cosθ, b sinθ); (a/cosθ, 0); and (0,b/sinθ).

$$p = (a/\cos\theta)\cos\theta' = (b/\sin\theta)\sin\theta'$$
$$p^2\cos^2\theta = a^2\cos^2\theta'$$
$$p^2\sin^2\theta = b^2\sin^2\theta'$$
$$p^2 = a^2\cos^2\theta' + b^2\sin^2\theta' = [(a^2+b^2)/2][\cos^2\theta' + \sin^2\theta'] + [(a^2-b^2)/2][\cos^2\theta' - \sin^2\theta']$$
$$= [(a^2+b^2)/2] + [(a^2-b^2)/2]\cos 2\theta'.$$

But $\quad 90° = \theta' + \alpha + \phi$

$\therefore \quad \cos 2\theta' = -\cos(2\alpha+2\phi)$

$\therefore \quad p^2 = [(a^2+b^2)/2] - [(a^2-b^2)/2]\cos(2\alpha+2\phi).$

Putting: $\quad p = d/K$

$$\left.\begin{aligned} s &= (a^2+b^2)/2 \\ t &= (a^2-b^2)/2 \end{aligned}\right\} \qquad (1)$$

$$d^2/K^2 = s - t.\cos(2\alpha+2\phi) \qquad (2)$$
$$= s - t.(\cos 2\alpha.\cos 2\phi - \sin 2\alpha.\sin 2\phi).$$

Referring d and α to specific sample planes by subscripts:

$$(d_2^2 - d_1^2)/K^2 = -t[(\cos 2\alpha_2 - \cos 2\alpha_1).\cos 2\phi - (\sin 2\alpha_2 - \sin 2\alpha_1).\sin 2\phi]$$
$$(d_3^2 - d_2^2)/K^2 = -t[(\cos 2\alpha_3 - \cos 2\alpha_2).\cos 2\phi - (\sin 2\alpha_3 - \sin 2\alpha_2).\sin 2\phi]$$
$$(d_2^2 - d_1^2)[(\cos 2\alpha_3 - \cos 2\alpha_2)\cos 2\phi - (\sin 2\alpha_3 - \sin 2\alpha_2)\sin 2\phi]$$
$$= (d_3^2 - d_2^2)[(\cos 2\alpha_2 - \cos 2\alpha_1)\cos 2\phi - (\sin 2\alpha_2 - \sin 2\alpha_1)\sin 2\phi]$$
$$\cos 2\phi([(d_2^2 - d_1^2)\cos 2\alpha_3 - (d_2^2 - d_1^2)\cos 2\alpha_2] - [(d_3^2 - d_2^2)\cos 2\alpha_2 + (d_3^2 - d_2^2)\cos 2\alpha_1])$$
$$= \sin 2\phi([(d_2^2 - d_1^2)\sin 2\alpha_3 - (d_2^2 - d_1^2)\sin 2\alpha_2] - [(d_3^2 - d_2^2)\sin 2\alpha_2 + (d_3^2 - d_2^2)\sin 2\alpha_1])$$
$$\tan 2\phi = \frac{(d_3^2 - d_2^2)\cos 2\alpha_1 + (d_1^2 - d_3^2)\cos 2\alpha_2 + (d_2^2 - d_1^2)\cos 2\alpha_3}{(d_3^2 - d_2^2)\sin 2\alpha_1 + (d_1^2 - d_3^2)\sin 2\alpha_2 + (d_2^2 - d_1^2)\sin 2\alpha_3}$$

This gives ∅, but note that according to the range of 2∅ used, it will refer to whichever of the principal ellipse axes lies in this range. Using the value 2∅ to put known values β = (2α+2∅) into (2) gives

$$d^2 = K^2 s + K^2 t\cos\beta \qquad (3)$$
$$(d_2^2 - d_1^2) = K^2 t(\cos\beta_2 - \cos\beta_1)$$
$$K^2 t = (d_2^2 - d_1^2)/(\cos\beta_2 - \cos\beta_1).$$

Knowing K^2t, using any equation (3), [K^2t.cosβ] can be evaluated and substituted to give:

$$K^2 s = K^2 t\cos\beta - d^2$$

From equations (1):

$$K^2a^2/2 + K^2b^2/2 = K^2 s$$
$$K^2a^2/2 - K^2b^2/2 = K^2 t.$$

Therefore:

$$K^2a^2 = (K^2s) + (K^2t)$$
$$K^2b^2 = (K^2s) - (K^2t).$$

Part 2. Combining Projection Ellipses

Consider each ellipse, with axial lengths 'Ka' and 'Kb' to be in its locally defined x* y* plane, with the axis length 'a' along x*. These axes can be related to the x, y, z, reference axes by a matrix **E** (see Appendix A) for which the elements are known. The strain can be represented by transformation matrix **S**. The distance 'd' to a tangent to the ellipse, measured in a perpendicular direction, at an angle ∅ to x*, can be written in terms of an axis rotated by ∅ from x* in the x* y* plane. The following equations can then be formed, using similar arguments to those in Appendix C:

$$((1/d) \;\; 0 \;\; 0) \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} = 1$$

$$\mathbf{x}^* = \mathbf{E}\mathbf{x}$$
$$\mathbf{x} = \mathbf{S}\mathbf{x}_0.$$

In the undeformed state:

$$(1/c \;\; 0 \;\; 0) \begin{pmatrix} \cos\phi_0 & \sin\phi_0 & 0 \\ -\sin\phi_0 & \cos\phi_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} \cos\phi_0 & -\sin\phi_0 & 0 \\ \sin\phi_0 & \cos\phi_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} (1/c) \\ 0 \\ 0 \end{pmatrix} = \frac{1}{c^2}$$

Hence:

$$((1/d) \quad 0 \quad 0)\begin{pmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\mathbf{ESS}^T\mathbf{E}^T \times \begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}(1/d)\\ 0\\ 0\end{pmatrix} = \frac{1}{c^2}.$$

Putting $\mathbf{T} = \mathbf{SS}^T$, and $\mathbf{M} = \mathbf{ETE}^T$, and multiplying gives:

$$(m_{11}/d^2)\cos^2\phi + (m_{12}/d^2)2\sin\phi\cos\phi + (m_{22}/d^2)\sin^2\phi = 1/c^2.$$

But:

$$(K^2a^2/d^2)\cos^2\phi + (K^2b^2/d^2)\sin^2\phi = 1$$

Hence, by comparison:

$$m_{11} = K^2a^2/c^2$$
$$m_{12} = 0$$
$$m_{22} = K^2b^2/c^2.$$

As $\mathbf{E}$ is known, these values can be substituted for these elements of $\mathbf{M}$ in the linear equations in the elements of $\mathbf{T}$ contained in $\mathbf{M} = \mathbf{E\,T\,E}^T$. This is the same matrix $\mathbf{T}$ as in Appendix C, where further procedure for solution is described.

APPENDIX C

Using the Eulerian system of Appendix A, the equation of a local sample plane is $z^* = 0$. That of a parallel plane (tangent to a strain ellipsoid) is $z^* = d$ or, in preparation for further matrix notation:

$$(0 \quad 0 \quad 1/d)\mathbf{x}^* = 1. \tag{1}$$

The local axes are related to the reference axes by:

$$\begin{pmatrix}x^*\\ y^*\\ z^*\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\beta & \sin\beta\\ 0 & -\sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} \tag{2}$$

where α and β for the particular sample are known. Combination of the above equations (1) and (2) gives, as the equation relating a tangent plane of the ellipsoid to the reference axes:

$$(\sin\alpha\sin\beta/d \quad -\cos\alpha\sin\beta/d \quad \cos\beta/d)\mathbf{x} = 1 \tag{3}$$

The deformation is specified by the following equation relating x, y, z coordinates of any point in the strained state to the coordinates of its initial position x_0, y_0, z_0:

$$\mathbf{x} = \mathbf{S}\mathbf{x}_0. \tag{4}$$

Hence [from equations (3) and (4)], points lying in a plane which was by the deformation to become a tangent plane at distance d, parallel to the sample, were specified by

$$(\sin\alpha\sin\beta/d \quad -\cos\alpha\sin\beta/d \quad \cos\beta/d)\mathbf{S}\mathbf{x}_0 = 1. \tag{5}$$

But this was a tangent to a sphere for which, if the distance from centre to tangent is denoted 'c', an equation could be written equivalent to (3) above:

$$(\sin\alpha_0\sin\beta_0/c \quad -\cos\alpha_0\sin\beta_0/c \quad \cos\beta_0/c)\mathbf{x}_0 = 1. \tag{6}$$

The angles α_0 and β_0 giving the prestrain orientation of the plane which was to become the sample are unknown, but using $\sin^2 + \cos^2 = 1$ we can write

$$(\sin\alpha_0\sin\beta_0/c \quad -\cos\alpha_0\sin\beta_0/c \quad \cos\beta_0/c) \times \begin{pmatrix}\sin\alpha_0 \ \sin\beta_0/c\\ -\cos\alpha_0 \ \sin\beta_0/c\\ \cos\beta_0/c\end{pmatrix} = \frac{1}{c^2}. \tag{7}$$

Using the equivalence of equations (5) and (6) above, we can therefore write:

$$(\sin\alpha\sin\beta/d \quad -\cos\alpha\sin\beta/d \quad \cos\beta/d)\mathbf{SS}^T \times \begin{pmatrix}\sin\alpha \ \sin\beta/d\\ -\cos\alpha \ \sin\beta/d\\ \cos\beta/d\end{pmatrix} = \frac{1}{c^2}. \tag{8}$$

Define a new matrix $\mathbf{T} = \mathbf{S\,S}^T$ and noting that it is symmetric, multiplying equation (8) out gives:

$$(\sin^2\alpha\sin^2\beta/d^2)t_{11} + (\cos^2\alpha\sin^2\beta/d^2)t_{22} + (\cos^2\beta/d^2)t_{33} + (-2\sin\alpha\cos\alpha\sin^2\beta/d^2)t_{12} + (2\sin\alpha\sin\beta\cos\beta/d^2)t_{13} + (-2\cos\alpha\sin\beta\cos\beta/d^2)t_{23} = 1/c^2. \tag{9}$$

In this equation, α, β and d are known from measurements for each sample plane. 'c' remains unknown, being the intersection object density on all planes in the unstrained state. Use of values of α, β and d for six (or more) orientations sets up simultaneous equations for which solution determines (or overdetermines) the elements 't' of the $\mathbf{T}$ matrix, giving them in terms of the common unknown 'c'.

Up to this point no assumption has been made about the strain, except that its description by a transformation matrix $\mathbf{S}$ precludes any translation of the origin. For convenience and convention, assume now that the strain is at constant volume and is irrotational. For constant volume det $\mathbf{S} = 1$ and so det $\mathbf{T} = 1$. This fixes the value of 'c' and leaves the matrix $\mathbf{T}$ specified uniquely. In terms of local coordinates on axes X, Y, Z paralleling the principal strains, an irrotational deformation can be represented as

$$\mathbf{X}' = \begin{pmatrix}l & 0 & 0\\ 0 & m & 0\\ 0 & 0 & n\end{pmatrix}\mathbf{X} \quad \text{or} \quad \mathbf{X}' = \mathbf{LX} \tag{10}$$

where l, m, n are the principal semiaxial lengths of the strain ellipsoid and $(l\,m\,n) = 1$. If the stationary principal strain axes of the irrotational strain represented by $\mathbf{S}$ are related to the reference axes by rotation $\mathbf{R}$, then

$$\mathbf{S} = \mathbf{R}^{-1}\mathbf{LR} \tag{11}$$

$$\mathbf{T} = \mathbf{SS}^T = \mathbf{R}^{-1}\mathbf{LRR}^T\mathbf{L}^T\mathbf{R}^{-1T}. \tag{12}$$

But, as $\mathbf{R}^{-1} = \mathbf{R}^T$ and $\mathbf{L}^T = \mathbf{L}$:

$$\mathbf{T} = \mathbf{R}^{-1}\mathbf{LRR}^{-1}\mathbf{LR} = \mathbf{R}^{-1}(\mathbf{L}.\mathbf{L})\mathbf{R}. \tag{13}$$

Determine the characteristic roots, λ ('eigenvalues'), of $\mathbf{T}$ from the cubic equation

$$\det(\mathbf{T} - \lambda\mathbf{I}) = 0 \tag{14}$$

where $\mathbf{I}$ is the identity (or unit) matrix. As these are also the roots of ($\mathbf{L.L}$) they are the principal quadratic extensions of the strain (λ_1, λ_2, λ_3).

The direction of each principal axis 'i' corresponding to each quadratic extension λ_i may be found by solving the following equation for the ratio of the coordinates of a vector along the principal axis (x_i, y_i, z_i):

$$\mathbf{T}\mathbf{x}_i - \lambda_i\mathbf{x}_i = 0. \tag{15}$$

These values can then be scaled such that $x_i^2 + y_i^2 + z_i^2 = 1$, and they are the direction cosines of that axis.

It is often geologically convenient to describe a strain in terms of the strike and dip of the XY plane and the pitch of X in that plane. These angles are conveniently, α, β, γ respectively using the Eulerian system as in Appendix A. To find them construct a matrix '$(\mathbf{R}^{-1})$' by compounding the principal axial direction cosines thus:

$$(\mathbf{R}^{-1}) = \begin{pmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ z_1 & z_2 & z_3\end{pmatrix} \tag{16}$$

Consider the matrix product $\mathbf{T}\,(\mathbf{R}^{-1})$, and then $\mathbf{T}\mathbf{x}_i = \lambda_i\mathbf{x}_i$ for its elements:

$$\mathbf{T}.(\mathbf{R}^{-1}) = \begin{pmatrix}(t_{11}x_1 + t_{12}y_1 + t_{13}z_1) & (t_{11}x_2 + t_{12}y_2 + t_{13}z_2) & \dots\\ (t_{21}x_1 + t_{22}y_1 + t_{23}z_1) & \dots & \\ \dots & \text{etc.} & \dots\end{pmatrix}$$

$$= \begin{pmatrix}\lambda_1x_1 & \lambda_2x_2 & \lambda_3x_3\\ \lambda_1x_1 & \lambda_2y_2 & \lambda_3y_3\\ \lambda_1z_1 & \lambda_2z_2 & \lambda_3z_3\end{pmatrix} = \begin{pmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ z_1 & z_2 & z_3\end{pmatrix}\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}$$

$$= (\mathbf{R}^{-1})(\mathbf{L}.\mathbf{L}) \tag{17}$$

If $\mathbf{TR}^{-1} = \mathbf{R}^{-1}\,\mathbf{L.L}$, then $\mathbf{T} = \mathbf{R}^{-1}\,\mathbf{L.L}\,\mathbf{R}$, and the matrix $\mathbf{R}$ (the inverse of $\mathbf{R}^{-1}$ defined in equation (16)) is shown to be the matrix $\mathbf{R}$ of equations (11)–(13) above. The angles required (α, β, γ) may be obtained from the identity of $\mathbf{R}$ with the matrix $\mathbf{E}$ of Appendix A, bearing in mind that for these matrices, their inverse is their transpose.

If, alternatively, the orientation of each axis is required in polar coordinates (bearing and plunge), these are simply obtained using the x, y, z coordinates (direction cosines) of the principal axes. For each, the bearing ø is given by arctan (y/x), and the plunge θ by arctan [$z/(x\cos ø + y\sin ø)$], using whichever value of ø between 0° and 360° gives a conventionally positive θ.